Table 2 $10^5\,\theta''$

Run No.	Exact Fit	Exact Estimate	Experimental Fit	Experimental Estimate
582	−0.48	−0.50	0.96	−0.50
575	−0.87	−0.83	0.44	−0.61
584	−1.63	−1.57	−2.02	−1.61
585	2.84	2.85	6.52	−0.69

with those from Eq. (2), however, a systematic overestimate of the exact α_{max} was observed for Run 585, which had the largest amplitude motion. This was felt to be caused by fitting the actual elliptic function solution with a sine wave. This bias can be eliminated through the approximation

$$sn\, u \doteq \sin y(1 + 4q \cos^2 y) \tag{6}$$

where $q = \exp(-\pi E_1' E_1^{-1})$, $E_1' = E_1(1 - k^2)^{1/2}$, $E_1 = E_1(k)$ complete elliptic integral of first kind, $k = -m(2 + m)^{-1}$, $m = 4K^2C_4C_3^{-1}$, and $u = (2E_1/\pi)y$. If $sn\, u$ is fitted by least squares to $A \sin y$, Eq. (6) yields

$$A = 1 + q \tag{7}$$

This correction for Run 585 is −2.4%. $3K^2$ and K^2 are each reduced by 4.8% in Figs. 2 and 3 and lines refitted, with the results $C_1 = -0.00602$, $C_2 = 0.000100$, $C_3 = 0.02074$, $C_4 = -0.0000380$. Thus, we see that the quasi-linear technique can yield the nonlinear damping coefficients to 5% and the cubic static moment coefficient to 2% from the exactly calculated points.

Analysis of Experimental Data

The actual flight data can now be fitted by Eqs. (2–4) with the results given in Tables 1 and 2. The standard error of the fits is quite good, but θ'' is now poorly predicted by Eq. (5). This probably means that C_3 and C_4 are not constants but must be allowed to be functions of Mach number,

$$\therefore \theta'' = [6C_4K^2\lambda + C_3' + 3C_4'K^2](2\theta')^{-1} \tag{8}$$

where $C_j' = (dC_j/dM)(dM/dx)$, M = Mach number. No provision for this effect is made in Chapman-Kirk analysis of these data although it could easily be incorporated by assuming a linear dependence of $C_{3,4}$ on Mach number and thereby increase the number of unknown coefficients of the differential equation to six.

The discrepancy shown in Fig. 1 can now be explained easily. The data contain both varying frequency and exponential damping. The fitted curve implicitly assumes these to be related by Eq. (5). Since the frequency variation dominates, the damping rate is given an erroneous bias. In Fig. 3 the quasi-linear damping is plotted for the experimental data and we see substantially different values of C_1 and C_2 are indicated. These values predict a limit cycle oscillation of amplitude 23° instead of the value of 15° implied by Ref. 1.

Figures 2 and 3 also show an additional advantage of the quasi-linear method. Namely, it gives an indication of the accuracy of the determination of the nonlinear coefficients since they appear as slopes of fitted lines. Indeed, we see that the nonlinear damping coefficient is poorly determined whereas the cubic static moment coefficient is very well determined.

Table 3 Quasi-linear results for exact input

	C_1	C_2	C_3	C_4
Exact	−0.00607	0.000105	0.02078	−0.0000375
Quasi-linear	−0.00600	0.000095	0.02072	−0.0000362
Quasi-linear[a]	−0.00602	0.000100	0.02074	−0.0000380

[a] Modified by Eq. (7).

References

[1] Chapman, G. T. and Kirk, D. B., "A New Method for Extracting Aerodynamic Coefficients from Free Flight Data," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 753–758.

[2] Murphy, C. H., "The Prediction of Nonlinear Pitching and Yawing Motion of Symmetric Missiles," *Journal of the Aeronautical Sciences*, Vol. 24, No. 7, July 1957, pp. 473–479.

[3] Murphy, C. H., "Quasi-Linear Analysis of the Nonlinear Motion of a Nonspinning Symmetric Missile," *Journal of Applied Mathematics and Physics (ZAMP)*, Vol. 14, Sept. 1963, pp. 630–643.

[4] Murphy, C. H., "Free Flight Motion of Symmetric Missiles," BRL Rept. 1216, AD442757, July 1963, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

[5] Chapman, G. T. and Kirk, D. B., private letter Oct. 15, 1969.

Reply by Authors to C. H. Murphy

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The quasi-linear method in the past has proven to be completely adequate in a great many instances; Dr. Murphy has certainly demonstrated its applicability in this example. However, there are problems for which the quasi-linear method would not be expected to give good results as, for example, when the test configuration departs greatly from being axisymmetric and the nonlinearities in the governing differential equations become large. Further, when using an approximate method on problems of this type, it is difficult to know a priori just how valid the solution is. Theoretical approximations interact in a variety of different ways with errors in the experimental data. The present method[1] eliminates this problem and, conceptually at least, can treat the complete differential equations involving six degrees of freedom.

Reference

[1] Chapman, G. T. and Kirk, D. B., "A Method for Extracting Aerodynamic Coefficients from Free-Flight Data," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 753–758.

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Reply by Author to M. N. Rao

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As pointed out by M. N. Rao in his comments[1] on our earlier studies[2,3] on classes of second- and third-order nonlinear systems, there indeed exists a definite relationship between the various nonlinear terms involved in the governing differential equation which permits it to be reducible to an equivalent linear differential equation (i.e., to an integrable form). However, a more general relationship than that put

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forth by Rao,[1] as applicable to a wider class of nonlinear systems, has been reported earlier by this author in his doctoral thesis[4] as well as in the open literature.[5] These studies not only give this generalized relationship but also put forth the concepts on which it is based and the rationale for obtaining such relationships.

Considering a general class of second-order, nonlinear systems in the form,

$$\ddot{x} + g(x)\dot{x}^2 + f(x)\dot{x} + F(x) = 0;\ f(x) \neq 0$$

(when $g(x) = 0$, i.e., without quadratic damping, this equation reduces to the case considered in Refs. 1 and 2), the sufficient condition for obtaining an equivalent linear system is that

$$F(x) = k_1 f(x) \left[\int_0^x f(\tau) \exp\left(\int_0^\tau g(\theta)\, d\theta\right) d\tau + k_2\right] \times \exp\left(-\int_0^x g(\theta)\, d\theta\right)$$

When $g(x) = 0$, this condition reduces to:

$$F(x) = k_1 f(x) \left(\int_0^x f(\theta)\, d\theta + k_2\right)$$

which essentially is the condition obtained by Rao.[1]

Regarding the two approaches suggested by Rao, although they prove to be interesting mathematical calisthenics, the example considered clearly shows that these approaches involve complexities in integration and algebra far more severe than those encountered in the earlier approach.[2–5] The transformation of inverting the role of dependent and independent variables used by Rao is quite well-known[6,7] and in fact has been in use to solve classes of Abel's nonlinear, first-order differential equations by converting them to linear, second-order equations, i.e., in the opposite direction. (In the partial differential equations domain, it has been known under the name of Hodograph transformations.[7])

For example, an Abel's equation of the type

$$\dot{x} = ax^2 + (bt + c)x^3$$

through a transformation of the type

$$x = \dot{y} = 1/dt/dy = 1/t'$$

can be converted to a linear second-order equation,

$$t'' + at' + bt + c = 0$$

However, in some cases these alternate approaches may prove to be useful and as such are welcome additions. The author wishes to take this opportunity to correct a mistake in Ref. 2 wherein Eq. (9) should read

$$\ddot{x} + ax\dot{x} + bx = 0$$

While this equation does correspond to an oscillatory system with nonlinear damping, it no longer represents surge tank oscillations.

References

[1] Rao, M.N., "Comments on Study of Nonlinear Systems," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1183–1184.

[2] Dasarathy, B. V. and Srinivasan, P., "Study of a Class of Nonlinear Systems," *AIAA Journal*, Vol. 6, No. 4, April 1968, pp. 736–737.

[3] Dasarathy, B. V. and Srinivasan, P., "Class of Nonlinear Third Order Systems Reducible to Equivalent Linear Systems," *AIAA Journal*, Vol. 6, No. 7, July 1968, pp. 1400–1401.

[4] Dasarathy, B. V., "Analytical Studies of Nonlinear, Non-Autonomous Systems," Ph.D. thesis, Indian Institute of Science, Bangalore, India, July 1968.

[5] Dasarathy, B. V., "Analysis of a Class of Nonlinear Systems," *Journal of Sound and Vibration*, Vol. 11, No. 1, Jan. 1970, pp. 139–144.

[6] Murphy, G. M., *Ordinary Differential Equations and Their Solutions*, Van Nostrand, New York, 1960.

[7] Ames, W. F., *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York, 1965.

Errata: "A Transformation Theory for the Compressible Turbulent Boundary Layer with Mass Transfer"

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SEVERAL errors occur in Equations (15–17) of this paper. The correct forms should read as follows:

$$\tilde{u} = (\bar{c}_f/2\bar{F})[\exp(\bar{F}R_{\bar{y}}) - 1];\ 0 \leq R_{\bar{y}} \leq R_{\bar{y}_s} \quad (15)$$

$$\ln(R_{\bar{\delta}}^2\bar{c}_f/2)^{1/2} = k_1\{(2/\bar{F})(\bar{c}_f/2)^{1/2} \times [(1 + 2\bar{F}/\bar{c}_f)^{1/2} - 1] - k_2 - 2\pi/k_1\} \quad (16)$$

$$R_{\bar{\theta}}/R_{\bar{\delta}} = (\bar{c}_f/2)^{1/2}(1 + 2\bar{F}/\bar{c}_f)^{1/2}(I_1 + 0.5\bar{F}I_3) - (\bar{c}_f/2)(1 + 2\bar{F}/\bar{c}_f)I_2 - 0.25\bar{F}(I_2 + 0.25\bar{F}I_4) \quad (17)$$

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